

# A certain minimization property implies a certain integrability

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## Abstract

The manifold  $M$  being compact and connected and  $H$  being a Tonelli Hamiltonian such that  $T^*M$  is equal to the dual tiered Mañé set, we prove that there is a partition of  $T^*M$  into invariant  $C^0$  Lagrangian graphs. Moreover, among these graphs, those that are  $C^1$  cover a dense  $G_\delta$  subset of  $T^*M$ . The dynamic restricted to each of these sets is non wandering.

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# 1 Introduction

In these article, we go on with our study of the so-called tiered Mañé set. We began this study in [1]. Let us recall that the dual tiered Mañé set  $\mathcal{N}_*^T(H)$  of a Tonelli Hamiltonian<sup>1</sup> is the union of all the dual Mañé sets of  $H$  associated to all the cohomology classes of  $M$ .

In [1], we proved that for a generic Tonelli Hamiltonian, the tiered Mañé set has no interior.

In our new article, we consider the following (non-generic) case : we assume that  $\mathcal{N}_*^T(H) = T^*M$ . In other words, we assume that every orbit of the Hamiltonian flow of  $H$  is globally minimizing for  $L - \lambda$ , where  $L$  is the Lagrangian associated to  $H$  and  $\lambda$  a closed 1-form (that depends on the considered orbit).

Such flows are part of a set of more general Tonelli Hamiltonian flows : those that have no conjugate points. For example, it is proved in [18] that any Anosov Hamiltonian level of a Tonelli Hamiltonian has no conjugate points. The same result for geodesic flows was proved in the 70's by W. Klingenberg in [11]. But the tiered Mañé set of an Anosov geodesic flow has no interior (see [1]) hence in this case, the dual tiered Mañé set is not equal to  $T^*M$ . In fact, we prove :

**Theorem 1** *Let  $M$  be a compact and connected manifold and let  $H : T^*M \rightarrow \mathbb{R}$  be a Tonelli Hamiltonian. Then the two following assertions are equivalent :*

1. *there exists a partition of  $T^*M$  into invariant Lipschitz Lagrangian graphs;*
2. *the dual tiered Mañé set of  $H$  is the whole cotangent bundle  $T^*M$ .*

Moreover, in this case :

- *there exists an invariant dense  $G_\delta$  subset  $\mathcal{G}$  of  $T^*M$  such that all the graphs of the partition that meets  $\mathcal{G}$  are in fact  $C^1$ .*
- *Mather's  $\beta$  function is everywhere differentiable.*

Let us emphasize why this result is surprising : we just ask that all the orbits are, in a certain way, minimizing, and we prove that they are well-distributed on invariant Lipschitz Lagrangian graphs.

An easy corollary is the following :

**Corollary 2** *Let  $M$  be a compact and connected manifold and let  $H : T^*M \rightarrow \mathbb{R}$  be a Tonelli Hamiltonian. Then the two following assertions are equivalent :*

1. *there exists a partition of  $T^*M$  into invariant Lipschitz Lagrangian graphs;*
2.  *$T^*M$  is covered by the union of its invariant Lipschitz Lagrangian graphs.*

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<sup>1</sup>all these notions will be precisely defined in next section

The same statement is true if we replace everywhere “Lipschitz” by “smooth”.

In [3], we proved a Birkhoff multidimensional theorem for Tonelli Hamiltonians. We deduce :

**Corollary 3** *Let  $M$  be a closed and connected manifold and let  $H : T^*M \rightarrow \mathbb{R}$  be a Tonelli Hamiltonian. Then the two following assertions are equivalent :*

1. *there exists a partition of  $T^*M$  into Lagrangian invariant smooth graphs;*
2.  *$T^*M$  is covered by the union of its Lagrangian invariant smooth submanifolds that are Hamiltonianly isotopic to some Lagrangian smooth graph.*

These results give us a characterization of a weak form of integrability; following [2], we say that a Tonelli Hamiltonian is  $C^0$ -integrable if there is a partition of  $T^*M$  into invariant  $C^0$ -Lagrangian graphs, one for each cohomology class in  $H^1(M, \mathbb{R})$ . We then prove that if all the orbits are in some Mañé set, then the Hamiltonian is  $C^0$ -integrable. A natural question is then :

**Question 1** : *does there exist any Tonelli Hamiltonian that is  $C^0$ -integrable but not  $C^1$ -integrable (i.e. for which the invariant graphs are not all  $C^1$ )?*

Let us notice that we finally prove that our hypotheses implies that the function  $\beta$  is everywhere differentiable. An interesting question, well-known from specialists, is : when the function  $\beta$  is everywhere differentiable, is the Hamiltonian  $C^0$ -integrable? In the case of closed surfaces, a positive answer to this question is given in [15].

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## 2 An overview of Mather-Mañé-Fathi theory of minimizing orbits

### 2.1 Tonelli Lagrangian and Hamiltonian functions

Let  $M$  be a compact and connected manifold endowed with a Riemannian metric . We denote a point of the tangent bundle  $TM$  by  $(q, v)$  with  $q \in M$  and  $v$  a vector tangent to  $M$  at  $q$ . The projection  $\pi : TM \rightarrow M$  is then  $(q, v) \rightarrow q$ . The notation  $(q, p)$  designates a point of the cotangent bundle  $T^*M$  with  $p \in T_q^*M$  and  $\pi^* : T^*M \rightarrow M$  is the canonical projection  $(q, p) \rightarrow q$ .

We consider a Lagrangian function  $L : TM \rightarrow \mathbb{R}$  which is  $C^2$  and:

- uniformly superlinear: uniformly on  $q \in M$ , we have:  $\lim_{\|v\| \rightarrow +\infty} \frac{L(q, v)}{\|v\|} = +\infty$ ;

- strictly convex: for all  $(q, v) \in TM$ ,  $\frac{\partial^2 L}{\partial v^2}(q, v)$  is positive definite.

Such a Lagrangian function will be called a *Tonelli Lagrangian function*.

We can associate to such a Lagrangian function the Legendre map  $\mathcal{L} = \mathcal{L}_L : TM \rightarrow T^*M$  defined by:  $\mathcal{L}(q, v) = \frac{\partial L}{\partial v}(q, v)$  which is a fibered  $C^2$  diffeomorphism and the Hamiltonian function  $H : T^*M \rightarrow \mathbb{R}$  defined by:  $H(q, p) = p(\mathcal{L}^{-1}(q, p)) - L(\mathcal{L}^{-1}(q, p))$  (such a Hamiltonian function will be called a *Tonelli Hamiltonian function*). The Hamiltonian function  $H$  is then superlinear, strictly convex in the fiber and  $C^2$ . We denote by  $(f_t^L)$  or  $(f_t)$  the Euler-Lagrange flow associated to  $L$  and  $(\varphi_t^H)$  or  $(\varphi_t)$  the Hamiltonian flow associated to  $H$ ; then we have :  $\varphi_t^H = \mathcal{L} \circ f_t^L \circ \mathcal{L}^{-1}$ .

If  $\lambda$  is a  $(C^\infty)$  closed 1-form of  $M$ , then the map  $T_\lambda : T^*M \rightarrow T^*M$  defined by :  $T_\lambda(q, p) = (q, p + \lambda(q))$  is a symplectic  $(C^\infty)$  diffeomorphism; therefore, we have :  $(\varphi_t^{H \circ T_\lambda}) = (T_\lambda^{-1} \circ \varphi_t \circ T_\lambda)$ , i.e. the Hamiltonian flow of  $H$  and  $H \circ T_\lambda$  are conjugated. Moreover, the Tonelli Hamiltonian function  $H \circ T_\lambda$  is associated to the Tonelli Lagrangian function  $L - \lambda$ , and it is well-known that :  $(f_t^L) = (f_t^{L-\lambda})$ ; the two Euler-Lagrange flows are equal. Let us emphasize that these flows are equal, but the Lagrangian functions, and then the Lagrangian actions differ and so the minimizing “objects” may be different.

## 2.2 Tiered sets : Mather, Aubry and Mañé

For a Tonelli Lagrangian function  $(L$  or  $L - \lambda)$ , J. Mather introduced in [17] (see [13] too) a particular subset  $\mathcal{A}(L - \lambda)$  of  $TM$  which he called the “static set” and which is now usually called the “*Aubry set*” (this name is due to A. Fathi)<sup>2</sup>. There exist different but equivalent definitions of this set (see [8] , [10], [13] and subsection 2.3) and it is known that two closed 1-forms which are in the same cohomological class define the same Aubry set :

$$[\lambda_1] = [\lambda_2] \in H^1(M) \Rightarrow \mathcal{A}(L - \lambda_1) = \mathcal{A}(L - \lambda_2).$$

We can then introduce the following notation : if  $c \in H^1(M)$  is a cohomological class,  $\mathcal{A}_c(L) = \mathcal{A}(L - \lambda)$  where  $\lambda$  is any closed 1-form belonging to  $c$ .  $\mathcal{A}_c(L)$  is compact, non empty and invariant under  $(f_t^L)$ . Moreover, J. Mather proved in [17] that it is a Lipschitz graph above a part of the zero-section (see [10] or subsection 2.3 too).

As we are as interested in the Hamiltonian dynamics as well as in the Lagrangian ones, let us define the dual Aubry set :

- if  $H$  is the Hamiltonian function associated to the Tonelli Lagrangian function  $L$ , its *dual Aubry set* is  $\mathcal{A}^*(H) = \mathcal{L}_L(\mathcal{A}(L))$ ;

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<sup>2</sup> These sets extend the notion of “Aubry-Mather” sets for the twist maps.

- if  $c \in H^1(M)$  is a cohomological class, then  $\mathcal{A}_c^*(H) = \mathcal{L}_L(\mathcal{A}_c(L))$  is the  $c$ -dual Aubry set; let us notice that for any closed 1-form  $\lambda$  belonging to  $c$ , we have :  $T_\lambda(\mathcal{A}^*(H \circ T_\lambda)) = \mathcal{A}_c^*(H)$ .

These sets are invariant under the Hamiltonian flow  $(\varphi_t^H)$ .

Another important invariant subset in the theory of Tonelli Lagrangian functions is the so-called Mather set. For it, there exists one definition (which is in [10], [13], [16] and subsection 2.4) : it is the closure of the union of the supports of the minimizing measures for  $L$ ; it is denoted by  $\mathcal{M}(L)$  and the *dual Mather set* is  $\mathcal{M}^*(H) = \mathcal{L}_L(\mathcal{M}(L))$  which is compact, non empty and invariant under the flow  $(\varphi_t^H)$ . As for the Aubry set, if  $c \in H^1(M)$  is a cohomological class, we define :  $\mathcal{M}_c(L) = \mathcal{M}(L - \lambda)$  which is independent of the choice of the closed 1-form  $\lambda$  belonging to  $c$ . Then  $\mathcal{M}_c^*(H) = \mathcal{L}_L(\mathcal{M}_c(L)) = T_\lambda(\mathcal{M}^*(H \circ T_\lambda))$  is invariant under  $(\varphi_t^H)$ ; we name it the  $c$ -dual Mather set.

In a similar way, if  $\mathcal{N}(L)$  is the Mañé set, the *dual Mañé set* is  $\mathcal{N}^*(H) = \mathcal{L}_L(\mathcal{N}(L))$ ; we note that if  $c \in H^1(M)$  and  $\lambda \in c$ , then  $\mathcal{N}_c(L) = \mathcal{N}(L - \lambda)$  is independent of the choice of  $\lambda \in c$  and then the  $c$ -dual Mañé set is  $\mathcal{N}_c^*(H) = \mathcal{L}_L(\mathcal{N}_c(L)) = T_\lambda(\mathcal{N}^*(H \circ T_\lambda))$ ; it is invariant under  $(\varphi_t^H)$ , compact and non empty but is not necessarily a graph.

For every cohomological class  $c \in H^1(M)$ , we have the inclusion :  $\mathcal{M}_c^*(H) \subset \mathcal{A}_c^*(H) \subset \mathcal{N}_c^*(H)$ . Moreover, there exists a real number denoted by  $\alpha_H(c)$  such that :  $\mathcal{N}_c^*(H) \subset H^{-1}(\alpha_H(c))$  (see [4] and [16]), i.e. each dual Mañé set is contained in an energy level. For  $c = 0$ , the value  $\alpha_H(0)$  is named the “critical value” of  $L$ .

DEFINITION. If  $H : T^*M \rightarrow \mathbb{R}$  is a Tonelli Hamiltonian function, the *tiered Aubry set*, the *tiered Mather set* and the *tiered Mañé set* are :

$$\mathcal{A}^T(L) = \bigcup_{c \in H^1(M)} \mathcal{A}_c(L); \quad \mathcal{M}^T(L) = \bigcup_{c \in H^1(M)} \mathcal{M}_c(L); \quad \mathcal{N}^T(L) = \bigcup_{c \in H^1(M)} \mathcal{N}_c(L).$$

Their dual sets are :

$$\mathcal{A}_*^T(H) = \bigcup_{c \in H^1(M)} \mathcal{A}_c^*(H); \quad \mathcal{M}_*^T(H) = \bigcup_{c \in H^1(M)} \mathcal{M}_c^*(H); \quad \mathcal{N}_*^T(H) = \bigcup_{c \in H^1(M)} \mathcal{N}_c^*(H).$$

### 2.3 Mañé potential, Peierls barrier, Aubry and Mañé sets

We gather in this sections some well-known results; the ones concerning the Peierls barrier are essentially due to A. Fathi (see [10]), the others concerning Mañé potential are given in [12], [6] and [7].

In the whole section,  $L$  is a Tonelli Lagrangian function.

NOTATIONS.

- given two points  $x$  and  $y$  in  $M$  and  $T > 0$ , we denote by  $\mathcal{C}_T(x, y)$  the set of absolutely continuous curves  $\gamma : [0, T] \rightarrow M$  with  $\gamma(0) = x$  and  $\gamma(T) = y$ ;
- the Lagrangian action along an absolutely continuous curve  $\gamma : [a, b] \rightarrow M$  is defined by :

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt;$$

- for each  $t > 0$ , we define the function  $h_t : M \times M \rightarrow \mathbb{R}$  by :  $h_t(x, y) = \inf\{A_{L+\alpha_H(0)}(\gamma); \gamma \in \mathcal{C}_t(x, y)\}$ ;
- the Peierls barrier is then the function  $h : M \times M \rightarrow \mathbb{R}$  defined by :

$$h(x, y) = \liminf_{t \rightarrow +\infty} h_t(x, y);$$

- we define the (Mañé) potential  $m : M \times M \rightarrow \mathbb{R}$  by :  $m(x, y) = \inf\{A_{L+\alpha_H(0)}(\gamma); \gamma \in \bigcup_{T>0} \mathcal{C}_T(x, y)\} = \inf\{h_t(x, y); t > 0\}$ .

Then, the Mañé potential verifies :

**Proposition 4** *We have :*

1.  $m$  is finite and  $m \leq h$ ;
2.  $\forall x, y, z \in M, m(x, z) \leq m(x, y) + m(y, z)$ ;
3.  $\forall x \in M, m(x, x) = 0$ ;
4. if  $x, y \in M$ , then  $m(x, y) + m(y, x) \geq 0$ ;
5. if  $M_1 = \sup\{L(x, v); \|v\| \leq 1\}$ , then :  $\forall x, y \in M, |m(x, y)| \leq (M_1 + \alpha_H(0))d(x, y)$ ;
6.  $m : M \times M \rightarrow \mathbb{R}$  is  $(M_1 + \alpha_H(0))$ -Lipschitz.

Now we can define :

DEFINITION.

- a absolutely continuous curve  $\gamma : I \rightarrow M$  defined on an interval  $I$  is a *ray* if :

$$\forall [a, b] \subset I, A_{L+\alpha_H(0)}(\gamma|_{[a, b]}) = h_{(b-a)}(\gamma(a), \gamma(b));$$

a ray is always a solution of the Euler-Lagrange equations;

- a absolutely continuous curve  $\gamma : I \rightarrow M$  defined on an interval  $I$  is *semistatic* if :

$$\forall [a, b] \subset I, m(\gamma(a), \gamma(b)) = A_{L+\alpha_H(0)}(\gamma|_{[a, b]});$$

a semistatic curve is always a ray;

- the *Mañé set* is then :  $\mathcal{N}(L) = \{v \in TM; \gamma_v \text{ is semistatic}\}$  where  $\gamma_v$  designates the solution  $\gamma_v : \mathbb{R} \rightarrow M$  of the Euler-Lagrange equations with initial condition  $v$  for  $t = 0$ ;  $\mathcal{N}(L)$  is contained in the critical energy level;
- a absolutely continuous curve  $\gamma : I \rightarrow M$  defined on an interval  $I$  is *static* if :

$$\forall [a, b] \subset I, -m(\gamma(b), \gamma(a)) = A_{L+\alpha_H(0)}(\gamma|_{[a,b]});$$

a static curve is always a semistatic curve;

- the *Aubry set* is then :  $\mathcal{A}(L) = \{v \in TM; \gamma_v \text{ is static}\}$ .

The following result is proved in [7] :

**Proposition 5** *If  $v \in TM$  is such that  $\gamma_v|_{[a,b]}$  is static for some  $a < b$ , then  $\gamma_v : \mathbb{R} \rightarrow M$  is static, i.e.  $v \in \mathcal{A}(L)$ .*

The Peierls barrier verifies (this proposition contains some results of [9], [10] and [5]) :

**Proposition 6** *(properties of the Peierls barrier  $h$ )*

1. *the values of the map  $h$  are finite and  $m \leq h$ ;*
2. *if  $M_1 = \sup\{L(x, v); \|v\| \leq 1\}$ , then :*

$$\forall x, y, x', y' \in M, |h(x, y) - h(x', y')| \leq (M_1 + \alpha_H(0))(d(x, x') + d(y, y'));$$

*therefore  $h$  is Lipschitz;*

3. *if  $x, y \in M$ , then  $h(x, y) + h(y, x) \geq 0$ ; we deduce :  $\forall x \in M, h(x, x) \geq 0$ ;*
4.  *$\forall x, y, z \in M, h(x, z) \leq h(x, y) + h(y, z)$ ;*
5.  *$\forall x \in M, \forall y \in \pi(\mathcal{A}(L)), m(x, y) = h(x, y)$  and  $m(y, x) = h(y, x)$ ;*
6.  *$\forall x \in M, h(x, x) = 0 \iff x \in \pi(\mathcal{A}(L))$ .*

The last item of this proposition gives us a characterization of the projected Aubry set  $\pi(\mathcal{A}(L))$ . Moreover, we have :

**Proposition 7** *(A. Fathi, [10], 6.3.3) When  $t$  tends to  $+\infty$ , uniformly on  $M \times M$ , the function  $h_t$  tends to the Peierls barrier  $h$ .*

A corollary of this result is given in [7] :

**Corollary 8** *([7], 4-10.9) All the rays defined on  $\mathbb{R}$  are semistatic.*

Let us give some properties of the Aubry and Mañé sets (see [13] and [6]) :

**Proposition 9** *Let  $L : TM \rightarrow \mathbb{R}$  be a Tonelli Lagrangian function. Then :*

- *the Aubry and Mañé set are compact, non empty and  $\mathcal{A}(L) \subset \mathcal{N}(L)$ ;*
- *the Aubry set is a Lipschitz graph above a part of the zero section;*
- *if  $\gamma : \mathbb{R} \rightarrow M$  is semistatic, then  $(\gamma, \dot{\gamma})$  is a Lipschitz graph above a part of the zero section;*
- *the  $\omega$  and  $\alpha$ -limit sets of every point of the Mañé set are contained in the Aubry set.*

Last item in proposition 6 gives us a criterion to some  $q \in M$  belong to some projected Aubry set. We will need a little more than this : we will need to know what happens for its lift, the Aubry set.

**Proposition 10** *Let  $c \in H^1(M)$  and  $\lambda \in c$ ,  $\varepsilon > 0$  and let  $L : TM \rightarrow \mathbb{R}$  be a Tonelli Lagrangian function. Then there exists  $T_0 > 0$  such that :*

$\forall T \geq T_0, \forall (q_0, v_0) \in \mathcal{A}_c(L), \forall \gamma : [0, T] \rightarrow M$  minimizing for  $L - \lambda$  between  $q_0$  and  $q_0$ , i.e. :

$\forall \eta : [0, T] \rightarrow M, \eta(0) = \eta(T) = q_0 \Rightarrow \int_0^T (L(\gamma, \dot{\gamma}) - \lambda(\dot{\gamma}) + \alpha_H(c)) \leq \int_0^T (L(\eta, \dot{\eta}) - \lambda(\dot{\eta}) + \alpha_H(c))$

then we have :  $d((q_0, v_0), (q_0, \gamma'(0))) \leq \varepsilon$

**PROOF** Let us assume that the result is not true; then we may find a sequence  $(T_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}_+^*$  tending to  $+\infty$ , a sequence  $\gamma_n : [0, T_n] \rightarrow M$  of absolutely continuous loops, all of whose minimizing for  $L - \lambda$  from  $q_n$  to  $q_n$  where  $(q_n, w_n) \in \mathcal{A}_c(L)$  such that the sequence  $(q_n, v_n) = (\gamma_n(0), \dot{\gamma}_n(0))$  satisfies :  $\forall n \in \mathbb{N}, d((q_n, v_n), (q_n, w_n)) \geq \varepsilon$ .

The sequence  $(q_n, v_n)$  is bounded (it is a consequence of the so-called “a priori compactness lemma” (see [10], corollary 4.3.2)); therefore we may extract a converging subsequence; we call it  $(q_n, v_n)$  again and  $(q_\infty, v_\infty)$  is its limit. Then  $q_\infty \in \pi(\mathcal{A}_c(L))$  because the Aubry set is closed. We denote by  $(q_\infty, w_\infty) \in \mathcal{A}_c(L)$  its lift. Then  $w_\infty = \lim_{n \rightarrow \infty} w_n$  because  $\mathcal{A}_c(L)$  is closed. Then :  $d((q_\infty, v_\infty), (q_\infty, w_\infty)) \geq \varepsilon$ .

Now we use proposition 7 : we know that if we define  $h_t^\lambda : M \times M \rightarrow \mathbb{R}$  by  $h_t^\lambda(x, y) = \inf \{A_{L-\lambda+\alpha_H(c)}(\gamma); \gamma \in \mathcal{C}_t(x, y)\}$  and  $h^\lambda(x, y) = \liminf_{t \rightarrow +\infty} h_t^\lambda(x, y)$ , the functions

$h_t^\lambda$  tend uniformly to  $h^\lambda$  when  $t$  tends to  $+\infty$ ; we have then :

$h_{T_n}^\lambda(q_n, q_n) = A_{L-\lambda+\alpha_H(c)}(\gamma_n)$  tends to  $h^\lambda(q_\infty, q_\infty) = 0$  when  $n$  tends to the infinite.

Let  $\gamma_\infty$  be the solution of the Euler-lagrange equations such that  $(\gamma_\infty(0), \dot{\gamma}_\infty(0)) = (q_\infty, v_\infty)$ . We want to prove that  $\gamma_\infty$  is static : we shall obtain a contradiction. When  $n$  is big enough,  $\gamma_n(T_n) = \gamma_n(0)$  is close to  $q_\infty$  and  $\gamma_n(1)$  is close to  $\gamma_\infty(1)$ . Let us fix  $\eta > 0$ ; then we define  $\Gamma_n^\eta : [0, T_n + 2\eta] \rightarrow M$  by :

- $\Gamma_{n|[0,1]}^\eta = \gamma_{\infty|[0,1]}$ ;



- $\Gamma_{n|[1,1+\eta]}^\eta$  is a short geodesic joining  $\gamma_\infty(1)$  to  $\gamma_n(1)$ ;
- $\forall t \in [1+\eta, T_n+\eta], \Gamma_n^\eta(t) = \gamma_n(t-\eta)$ ;
- $\Gamma_{n|[T_n+\eta, T_n+2\eta]}^\eta$  is a short geodesic joining  $\gamma_n(T_n)$  to  $\gamma_\infty(0)$ .

If we choose carefully a sequence  $(\eta_n)$  tending to 0, we have :

$$\lim_{n \rightarrow \infty} A_{L-\lambda+\alpha_H(c)}(\Gamma_n^{\eta_n}) = \lim_{n \rightarrow \infty} A_{L-\lambda+\alpha_H(c)}(\gamma_n) = 0.$$

Because the contribution to the action of the two small geodesic arcs tends to zero (if the  $\eta_n$  are well chosen), this implies :

$$A_{L-\lambda+\alpha_H(c)}(\gamma_\infty|_{[0,1]}) + m^\lambda(\gamma_\infty(1), \gamma_\infty(0)) \leq 0,$$

where  $m^\lambda$  designates Mañé potential for the Lagrangian function  $L-\lambda$ . We deduce then from the definition of Mañé potential that  $m^\lambda(\gamma_\infty(0), \gamma_\infty(1)) + m^\lambda(\gamma_\infty(1), \gamma_\infty(0)) = 0$  and that :  $A_{L-\lambda+\alpha_H(c)}(\gamma_\infty|_{[0,1]}) = m^\lambda(\gamma_\infty(0), \gamma_\infty(1))$ . It implies then that  $A_{L-\lambda+\alpha_H(c)}(\gamma_\infty|_{[0,1]}) = -m^\lambda(\gamma_\infty(1), \gamma_\infty(0))$ . Let us notice that, changing slightly  $\Gamma_n^\eta$ , we obtain too :

$$\forall [a, b] \subset [0, +\infty[, A_{L-\lambda+\alpha_H(c)}(\gamma_\infty|_{[a,b]}) = -m^\lambda(\gamma_\infty(b), \gamma_\infty(a));$$

therefore  $\gamma_\infty|_{[0,+\infty[}$  is static. To conclude, we use proposition 5. □

## 2.4 Minimizing measures, Mather $\alpha$ and $\beta$ functions

The general references for this section are [16] and [15]. Let  $\mathfrak{M}(L)$  be the space of compactly supported Borel probability measures invariant under the Euler-Lagrange flow  $(f_t^L)$ . To every  $\mu \in \mathfrak{M}(L)$  we may associate its average action  $A_L(\mu) = \int_{TM} L d\mu$ . It is proved in [16] that for every  $f \in C^1(M, \mathbb{R})$ , we have :  $\int df(q).vd\mu(q, v) = 0$ . Therefore we can define on  $H^1(M, \mathbb{R})$  a linear functional  $\ell(\mu)$  by :  $\ell(\mu)([\lambda]) = \int \lambda(q).vd\mu(q, v)$  (here  $\lambda$  designates any closed 1-form). Then there exists a unique element  $\rho(\mu) \in H_1(M, \mathbb{R})$  such that :

$$\forall \lambda, \int_{TM} \lambda(q).vd\mu(q, v) = [\lambda].\rho(\mu).$$

The homology class  $\rho(\mu)$  is called the *rotation vector* of  $\mu$ . Then the map  $\mu \in \mathfrak{M}(L) \rightarrow \rho(\mu) \in H_1(M, \mathbb{R})$  is onto. We can then define Mather  $\beta$ -function  $\beta : H_1(M, \mathbb{R}) \rightarrow \mathbb{R}$  that associates the minimal value of the average action  $A_L$  over the set of measures of  $\mathfrak{M}(L)$  with rotation vector  $h$  to each homology class  $h \in H_1(M, \mathbb{R})$ . We have :

$$\beta(h) = \min_{\mu \in \mathfrak{M}(L); \rho(\mu)=h} A_L(\mu).$$

A measure  $\mu \in \mathfrak{M}(L)$  realizing such a minimum, i.e. such that  $A_L(\mu) = \beta(\rho(\mu))$  is called a *minimizing measure with rotation vector*  $\rho(\mu)$ . The  $\beta$  function is convex

and superlinear, and we can define its conjugate function (given by Fenchel duality)  $\alpha : H^1(M, \mathbb{R}) \rightarrow \mathbb{R}$  by :

$$\alpha([\lambda]) = \max_{h \in H_1(M, \mathbb{R})} ([\lambda].h - \beta(h)) = - \min_{\mu \in \mathfrak{M}(L)} A_{L-\lambda}(\mu).$$

A measure  $\mu \in \mathfrak{M}(L)$  realizing the minimum of  $A_{L-\lambda}$  is called a  $[\lambda]$ -*minimizing measure*. Being convex, Mather's  $\beta$  function has a subderivative at any point  $h \in H_1(M, \mathbb{R})$ ; i.e. there exists  $c \in H^1(M, \mathbb{R})$  such that :  $\forall k \in H_1(M, \mathbb{R}), \beta(h) + c.(k - h) \leq \beta(k)$ . We denote by  $\partial\beta(h)$  the set of all the subderivatives of  $\beta$  at  $h$ . By Fenchel duality, we have :  $c \in \partial\beta(h) \Leftrightarrow c.h = \alpha(c) + \beta(h)$ .

Then we introduce the following notations :

- if  $h \in H_1(M, \mathbb{R})$ , the Mather set for the rotation vector  $h$  is :

$$\mathcal{M}^h(L) = \bigcup \{ \text{supp} \mu; \mu \text{ is minimizing with rotation vector } h \};$$

- if  $c \in H^1(M, \mathbb{R})$ , the Mather set for the cohomology class  $c$  is :

$$\mathcal{M}_c(L) = \bigcup \{ \text{supp} \mu; \mu \text{ is } c\text{-minimizing} \}.$$

The following equivalences are proved in [15] for any pair  $(h, c) \in H_1(M, \mathbb{R}) \times H^1(M, \mathbb{R})$  :

$$\mathcal{M}^h(L) \cap \mathcal{M}_c(L) \neq \emptyset \Leftrightarrow \mathcal{M}^h(L) \subset \mathcal{M}_c(L) \Leftrightarrow c \in \partial\beta(h).$$

As explained in subsection 2.2, the dual Mather set for the cohomology class  $c$  is defined by :  $\mathcal{M}_c^*(H) = \mathcal{L}_L(\mathcal{M}_c(L))$ . If  $\mathcal{M}^*(\mathcal{H})$  designates the set of compactly supported Borel probability measures of  $T^*M$  that are invariant by the Hamiltonian flow  $(\varphi_t)$ , then the map  $\mathcal{L}_* : \mathfrak{M}(L) \rightarrow \mathfrak{M}^*(H)$  that push forward the measures by  $\mathcal{L}$  is a bijection. We denote  $\mathcal{L}_*(\mu)$  by  $\mu^*$  and say that the measures are dual. We say too that  $\mu^*$  is minimizing if  $\mu$  is minimizing in the previous sense.

Moreover, the Mather set  $\mathcal{M}_c^*(H)$  is a subset of the Mañé set  $\mathcal{N}_c^*(H)$  and every invariant Borel probability measure the support of whose is in  $\mathcal{N}_c^*(H)$  is  $c$ -minimizing.

## 2.5 The link with the weak KAM theory

If  $\lambda$  is a closed 1-form on  $M$ , we can consider the Lax-Oleinik semi-groups of  $L - \lambda$ , defined on  $C^0(M, \mathbb{R})$  by :

- the negative one :  $T_t^{\lambda, -} u = \min \left( u(\gamma(0)) + \int_0^t (L(\gamma(s)) - \lambda(\gamma(s)) \dot{\gamma}(s)) ds \right);$   
where the infimum is taken on the set of  $C^1$  curves  $\gamma : [0, t] \rightarrow M$  such that  $\gamma(t) = q$ ;

- the positive one :  $T_t^{\lambda,+}u(q) = \max \left( u(\gamma(t)) - \int_0^t (L(\gamma(s), \dot{\gamma}(s)) - \lambda(\gamma(s)) \cdot \dot{\gamma}(s)) ds \right)$  ;  
where the infimum is taken on the set of  $C^1$  curves  $\gamma : [0, t] \rightarrow M$  such that  $\gamma(0) = q$ .

A. Fathi proved in [10] that for each closed 1-form  $\lambda$ , there exists  $k \in \mathbb{R}$  and  $u \in C^0(M, \mathbb{R})$  such that :  $\forall t > 0, T_t^{\lambda,-}u = u - kt$  (resp.  $\forall t > 0, T_t^{\lambda,+}u = u + kt$ ). In this case, we have :  $k = \alpha([\lambda])$ . The function  $u$  is called a negative (resp. positive) weak KAM solution for  $L - \lambda$ . We denote the set of negative (resp. positive) weak KAM solutions for  $L - \lambda$  by  $\mathcal{S}_\lambda^-$  (resp.  $\mathcal{S}_\lambda^+$ ).

Moreover, it is proved too that a function  $u : M \rightarrow \mathbb{R}$  that is  $C^1$  is a positive weak KAM solution if and only if it is a negative weak KAM solution if and only if it is a solution of the Hamilton-Jacobi equation :  $H(q, \lambda(q) + du(q)) = \alpha([\lambda])$ . It is equivalent too to the fact that the graph of  $\lambda + du$  is invariant by the Hamiltonian flow  $(\varphi_t^H)$ .

But in general, the weak KAM solutions are not  $C^1$  and the graph of  $\lambda + du$  is not invariant by the Hamiltonian flow. There is an invariant subset contained in all these graphs : the dual Aubry set. Let us now recall which characterization of this set is given by A. Fathi in [10].

A pair  $(u_-, u_+)$  of negative-positive weak KAM solution is called a pair of conjugate weak KAM solutions if  $u_-|_{\pi(\mathcal{M}(L-\lambda))} = u_+|_{\pi(\mathcal{M}(L-\lambda))}$ . Each negative weak KAM solution has an unique conjugate positive weak KAM solution, and we define for any pair  $(u_-, u_+) \in \mathcal{S}_\lambda^- \times \mathcal{S}_\lambda^+$  of conjugate weak KAM solutions for  $L - \lambda$  :

- $\mathcal{I}(u_-, u_+) = \{q \in M, u_-(q) = u_+(q)\}$ ;
- $\tilde{\mathcal{I}}(u_-, u_+) = \{(q, du_-(q)); q \in \mathcal{I}(u_-, u_+)\} = \{(q, du_+(q)); q \in \mathcal{I}(u_-, u_+)\}$ .

Then :  $\mathcal{A}_{[\lambda]}^*(H) = T_\lambda(\bigcap \tilde{\mathcal{I}}(u_-, u_+))$  where the intersection is taken on all the pairs of conjugate weak KAM solutions for  $L - \lambda$ . Moreover :  $\mathcal{N}_{[\lambda]}^*(H) = T_\lambda(\bigcup \tilde{\mathcal{I}}(u_-, u_+))$  where the union is taken on all the pairs of conjugate weak KAM solutions for  $L - \lambda$ .

An immediate corollary of all these results is the following : if  $\pi^*(\mathcal{A}_{[\lambda]}^*(H)) = M$ , then there is a unique negative weak KAM solution  $u$  and a unique positive weak KAM solution for  $L - \lambda$ , they are equal and  $C^{1,1}$  (i.e.  $C^1$  with a Lipschitz derivative). In this case, we have :  $\mathcal{A}_{[\lambda]}^*(H) = \mathcal{N}_{[\lambda]}^*(H)$  is the graph of  $\lambda + du$ .

### 3 Proof of theorem 1

We assume that  $H$  is a Tonelli Hamiltonian such that  $\mathcal{N}_*^T(H) = T^*M$ .

In order to prove theorem 1, we begin by proving that the periodic orbits are on some invariant totally periodic Lagrangian graphs :

**Proposition 11** *For every closed 1-form  $\lambda$  of  $M$ , for every  $(q_0, p_0) \in T^*M$  that is  $T$ -periodic for a certain  $T > 0$  and whose orbit under the Hamiltonian flow is minimizing for  $L - \lambda$ , then  $(q_0, p_0)$  belongs to a  $C^1$  invariant Lagrangian graph  $\mathcal{T}$  such that the orbit of every element of  $\mathcal{T}$  is  $T$ -periodic, homotopic to the one of  $(q_0, p_0)$  and has the same action for the Lagrangian  $L - \lambda$  as the orbit of  $(q, p)$ . Moreover,  $\mathcal{T}$  is the graph of a closed 1-form that has the same cohomology class as  $\lambda$ .*

PROOF Let us consider  $(q_0, p_0)$  as in the statement. Then, if we denote the cohomology class of  $\lambda$  by  $[\lambda]$ , we have :  $(q_0, p_0) \in \mathcal{N}_{[\lambda]}^*(H)$ , i.e.  $(q_0, p_0)$  belongs to the Mañé set associated to the cohomology class of  $\lambda$ . Let us use the notation :  $\gamma_0(t) = \pi \circ \varphi_t(q_0, p_0)$ .

Because of Tonelli theorem, we know that for every  $q \in M$ , there exists a piece of orbit  $(\varphi_t(q, p))_{t \in [0, T]}$  such that, if we denote the projection of this piece of orbit by  $\gamma_q$  (i.e.  $\gamma_q(t) = \pi \circ \varphi_t(q, p)$ ), then we have :

- $\gamma_q(T) = \gamma_q(0) = q$ ;
- $\gamma_q$  is homotopic to  $\gamma_0$ ;
- for every absolutely continuous arc  $\eta : [0, T] \rightarrow M$  that is homotopic to  $\gamma_0$  and such that :  $\eta(0) = \eta(T) = q$ , we have :  $\int_0^T (L(\gamma_q, \dot{\gamma}_q) - \lambda(\dot{\gamma}_q)) \leq \int_0^T (L(\eta, \dot{\eta}) - \lambda(\dot{\eta}))$ .

As every point on  $T^*M$  is in some Mañé set, then the orbit of every point has to be a graph by proposition 9. We deduce that :  $\varphi_T(q, p) = (q, p)$ , hence  $(q, p)$  is a  $T$ -periodic point. It defines an invariant probability measure  $\mu_q$ , the one equidistributed along this orbit, defined by :

$$\forall f \in C^0(T^*M, \mathbb{R}), \int f d\mu = \frac{1}{T} \int_0^T f \circ \varphi_t(q, p) dt.$$

As the support of this measure is in some Mañé set, this measure is minimizing for  $L + \nu$  where  $\nu$  is some closed 1-form. The rotation vector of this measure is  $\frac{1}{T}[\gamma] = \frac{1}{T}[\gamma_0]$  where we denote the homology class of  $\gamma$  by  $[\gamma]$ ; hence, having the same rotation vector, the supports of the measures  $\mu_q$  and  $\mu_{q_0}$  belong to the same Mather set and the support of  $\mu_q$  is in  $\mathcal{N}_{[\lambda]}^*(H)$ . We deduce that :

$$\forall q \in M, -T\alpha([\lambda]) = \int_0^T (L(\gamma, \dot{\gamma}) - \lambda(\dot{\gamma})) = \int_0^T (L(\gamma_0, \dot{\gamma}_0) - \lambda(\dot{\gamma}_0))$$

because all these measures are minimizing for  $L - \lambda$ .

Finally, for all  $q \in M$ , we have found a point  $(q, p)$  that is in the Mather set  $\mathcal{M}_{[\lambda]}^*(H)$ . As the Mather set is a Lipschitz graph, then the set of these points  $(q, p)$  is a Lipschitz graph and coincides with the Mather set  $\mathcal{M}_{[\lambda]}^*(H)$ . Moreover, we know that the Aubry set is a graph that contains the Mather set. Hence  $\mathcal{A}_{[\lambda]}^*(H) = \mathcal{M}_{[\lambda]}^*(H)$ .

In this case, this set is the graph of a Lipschitz closed 1-form whose cohomology class is  $[\lambda]$  (see subsection 2.5). As the dynamic restricted to this  $C^0$ -Lagrangian graph is totally periodic, i.e. as  $\varphi_{T|_{\mathcal{A}_{[\lambda]}^*(H)}} = \text{Id}_{\mathcal{A}_{[\lambda]}^*(H)}$ , we know that this graph is in fact  $C^1$  (this is proved in [2] by way of the so-called Green bundles).

□

We can apply this proposition to every periodic orbit. Indeed, such a periodic orbit is always contained in some Mañé set  $\mathcal{N}_c^*(H)$ . We deduce from the previous proposition that  $\mathcal{A}_c^*(H)$  is a  $C^1$  Lagrangian graph, and that all the orbits contained in  $\mathcal{A}_c^*(H)$  are periodic with the same period and are homotopic to each other. Moreover, we have seen in subsection 2.5 that when the Aubry set is a graph above the whole zero section, then it coincides with the Mañé set. Hence, we have proved that  $\mathcal{N}_c^*(H)$  is a  $C^1$  Lagrangian graph, and that all the orbits contained in  $\mathcal{N}_c^*(H)$  are periodic with the same period and are homotopic to each other.

Let us now explain what happens to the other Mañé sets, that correspond to the other cohomology classes.

**Proposition 12** *For every cohomology class  $c \in H^1(M, \mathbb{R})$ , we have :  $\mathcal{A}_c^*(H) = \mathcal{N}_c^*(H)$  is the graph  $\mathcal{G}_c$  of a Lipschitz closed 1-form.*

PROOF Let us assume that  $(q, p) \in \mathcal{A}_c^*(H)$ . Let  $\lambda$  be a closed 1-form such that  $[\lambda] = c$ . Then there exists a sequence  $(T_n)$  tending to  $+\infty$  and a sequence  $(\gamma_n)$  of absolutely continuous arcs  $\gamma_n : [0, T_n] \rightarrow M$  that are minimizing, such that  $\gamma(0) = \gamma(T_n) = q$  and such that :  $\lim_{n \rightarrow \infty} \int_0^{T_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \lambda(\dot{\gamma}_n(t)) + \alpha(c)) dt = 0$  where  $\alpha$  designates the  $\alpha$  function of Mather. As every  $\gamma_n$  is minimizing, it is the projection of a piece of orbit :  $\gamma_n(t) = \pi \circ \varphi_t(q, p_n)$ . The corresponding orbit, being in a certain Mañé set, has to be a graph, hence it is periodic :  $\varphi_{t_n}(q, p_n) = (q, p_n)$ . Moreover, we know (see proposition 10) that in this case :  $\lim_{n \rightarrow \infty} (q, p_n) = (q, p)$ .

We can use proposition 11. Let  $c_n \in H^1(M, \mathbb{R})$  be the cohomology class such that  $(q, p_n) \in \mathcal{N}_{c_n}^*(H)$ . Then there exists a closed 1-form  $\lambda_n$ , whose cohomology class is  $c_n$ , so that  $\mathcal{N}_{c_n}^*(H)$  is the graph of  $\lambda_n$ . We have in particular :  $p_n = \lambda_n(q)$  and  $p = \lim_{n \rightarrow \infty} \lambda_n(q)$ . Let us now prove that for every  $Q \in M$ , the sequence  $(Q, \lambda_n(Q))$  converges to some point  $(Q, P)$  that belongs to  $\mathcal{A}_c^*(H)$ . We will deduce that  $\mathcal{A}_c^*(H) = \mathcal{N}_c^*(H)$  is the graph of a Lipschitz closed 1-form and then the proposition.

So let us consider  $Q \in M$ . For every  $n \in \mathbb{N}$ , we know by proposition 11 that  $(Q, \lambda_n(Q))$  is  $t_n$ -periodic and that if we denote the projection of its orbit by  $\Gamma_n(t) = \pi \circ \varphi_t(Q, \lambda_n(Q))$ , then we have :

- $\Gamma_n$  is homotopic to  $\gamma_n$ ;

- $\int_0^{t_n} (L(\Gamma_n(t), \dot{\Gamma}_n(t)) - \lambda_n(\dot{\Gamma}_n(t)))dt = \int_0^{t_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \lambda_n(\dot{\gamma}_n(t)))dt.$

We can then compute (the notation  $[\lambda][\gamma]$  is just the usual product of a cohomology class with a homology class) :

$$\begin{aligned} & \int_0^{t_n} (L(\Gamma_n(t), \dot{\Gamma}_n(t)) - \lambda(\dot{\Gamma}_n(t)) + \alpha(c))dt = \\ & \int_0^{t_n} (L(\Gamma_n(t), \dot{\Gamma}_n(t)) - \lambda_n(\dot{\Gamma}_n(t)))dt - [\lambda - \lambda_n][\Gamma_n] + \alpha(c)t_n = \\ & \int_0^{t_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \lambda_n(\dot{\gamma}_n(t)))dt - [\lambda - \lambda_n][\gamma_n] + \alpha(c)t_n = \\ & \int_0^{t_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \lambda(\dot{\gamma}_n(t)) + \alpha(c))dt. \end{aligned}$$

Then :  $\lim_{n \rightarrow \infty} \int_0^{t_n} (L(\Gamma_n(t), \dot{\Gamma}_n(t)) - \lambda(\dot{\Gamma}_n(t)) + \alpha(c))dt = 0$ . By proposition 10, this implies that  $Q$  belongs to the projected Aubry set  $\pi(\mathcal{A}_c^*(H))$  and that the sequence  $(Q, \lambda_n(Q))$  converges to the unique point of  $\mathcal{A}_c^*(H)$  that is above  $Q$ .

□

**Proposition 13** *With the previous notations, the graphs  $\mathcal{G}_c$  are disjoint :*

$$\forall c, d \in H^1(M, \mathbb{R}), c \neq d \Rightarrow \mathcal{G}_c \cap \mathcal{G}_d = \emptyset.$$

PROOF We borrow the main elements of the proof to [14]. Let us assume that there exists  $c, d \in H^1(M, \mathbb{R})$  such that  $\mathcal{G}_c \cap \mathcal{G}_d \neq \emptyset$ . Then  $\mathcal{G}_c \cap \mathcal{G}_d$  is a compact invariant subset and there exists an invariant Borel probability measure  $\mu^*$  (dual of  $\mu$ ) whose support is contained in  $\mathcal{G}_c \cap \mathcal{G}_d$ . Hence  $\mu$  is minimizing for  $L - \lambda$  and  $L - \eta$  if  $[\lambda] = c$  and  $[\eta] = d$  :

$$\int (L - \lambda + \alpha(c))d\mu = 0 \quad \text{and} \quad \int (L - \eta + \alpha(d))d\mu = 0.$$

We deduce that for every  $t \in [0, 1]$ , we have :

$$\int (L - (t\lambda + (1-t)\eta) + t\alpha(c) + (1-t)\alpha(d))d\mu = 0$$

and then :  $\alpha(tc + (1-t)d) \geq - \int (L - (t\lambda + (1-t)\eta))d\mu = t\alpha(c) + (1-t)\alpha(d)$ . As the function  $\alpha$  is convex, this implies :  $\alpha(tc + (1-t)d) = t\alpha(c) + (1-t)\alpha(d)$ . Hence  $\mu$  is minimizing for  $L - (t\lambda + (1-t)\eta)$ . This implies that the support of  $\mu^*$  is contained in  $\mathcal{M}_{tc+(1-d)}^*(H) \subset \mathcal{A}_{tc+(1-d)}^*(H) = \mathcal{N}_{tc+(1-d)}^*(H) = \mathcal{G}_{tc+(1-d)}$ . Let us now consider  $(q, p) \in \mathcal{G}_{\frac{1}{2}(c+d)}$ . As  $(q, p)$  belongs to  $\mathcal{A}_{\frac{1}{2}(c+d)}^*(H)$ , there exists a sequence  $(T_n)$  tending to  $+\infty$  and a sequence of  $C^1$  arcs  $\gamma_n : [0, T_n] \rightarrow M$  such that  $\gamma_n(0) = \gamma_n(T_n) = q$  and :

$$\lim_{n \rightarrow \infty} \left( \int_0^{T_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \frac{1}{2}(\lambda + \eta)(\dot{\gamma}_n(t)) + \alpha(\frac{1}{2}(c+d)))dt \right) = 0.$$

The left term of the previous equality is the limit of the sum of two terms :

$\frac{1}{2} \int_0^{T_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \lambda(\dot{\gamma}_n(t)) + \alpha(c))dt$  and  $\frac{1}{2} \int_0^{T_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \eta(\dot{\gamma}_n(t)) + \alpha(d))dt$ , each of these terms being non negative. We deduce that :

- $\lim_{n \rightarrow \infty} \int_0^{T_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \lambda(\dot{\gamma}_n(t)) + \alpha(c)) dt = 0;$
- $\lim_{n \rightarrow \infty} \int_0^{T_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \eta(\dot{\gamma}_n(t)) + \alpha(d)) dt = 0;$

and by proposition 10 :

$$\lim_{n \rightarrow \infty} (\gamma_n(0), \dot{\gamma}_n(0)) \in \mathcal{A}_c(H) \cap \mathcal{A}_d(H).$$

We have finally proved that  $\mathcal{G}_{\frac{1}{2}(c+d)} = \mathcal{A}_{\frac{1}{2}(c+d)}^*(H) \subset \mathcal{A}_c^*(H) \cap \mathcal{A}_d^*(H) = \mathcal{G}_c \cap \mathcal{G}_d$ , hence the two graphs  $\mathcal{G}_c$  and  $\mathcal{G}_d$  are equal, and their cohomology classes are also equal :  $c = d$ .  $\square$

Let us now finish the proof of theorem 1. We have found a partition of  $T^*M$  into Lipschitz Lagrangian graphs  $(\mathcal{G}_c)_{c \in H^1(M, \mathbb{R})}$ , where  $\mathcal{G}_c$  is the graph of a Lipschitz 1-form whose cohomology class is  $c$  and is equal to  $\mathcal{A}_c^*(H) = \mathcal{N}_c^*(H)$ . Each Mañé set being chain recurrent, we deduce that the dynamic restricted to each  $\mathcal{G}_c$  is chain recurrent. We are then exactly in the case of a  $C^0$ -integrable Hamiltonian that we described in [2]. We can apply the results of [2] and deduce that there exists a dense  $G_\delta$ -subset of  $T^*M$  filled by invariant  $C^1$  Lagrangian graphs. Finally, let us notice that it is proved in [15] that the  $\beta$  function of every  $C^0$  integrable Tonelli Hamiltonian is differentiable everywhere. This ends the proof of the implication :  $2 \Rightarrow 1$ .

Let us now prove that  $1 \Rightarrow 2$ . We assume that there is a partition of  $T^*M$  into invariant Lagrangian Lipschitz graph. Then to each of these Lipschitz graphs corresponds a  $C^{1,1}$  weak KAM solution and then the orbit of every point of this graph is in some Mañé set. This implies :  $T^*M = \mathcal{N}_*^T(M)$ .

## 4 Proof of the corollaries

### 4.1 Proof of corollary 2

We only have to prove that  $2 \Rightarrow 1$ . We assume that  $T^*M$  is covered by the union of the invariant Lipschitz Lagrangian graphs (resp. smooth Lagrangian graphs). Then to each of these Lipschitz graphs correspond a  $C^{1,1}$  weak KAM solution and then the orbit of every point of this graph is in some Mañé set. This implies :  $T^*M = \mathcal{N}_*^T(H)$ . We can apply theorem 1 and proposition 12. Then there exists a partition of  $T^*M$  into Lipschitz Lagrangian graphs  $(\mathcal{G}_c)_{c \in H^1(M, \mathbb{R})}$ , where  $\mathcal{G}_c$  is the graph of a Lipschitz 1-form whose cohomology class is  $c$  and is equal to  $\mathcal{A}_c^*(H) = \mathcal{N}_c^*(H)$ . Let us look at what happens in the smooth case : if  $N$  is one of the smooth invariant Lagrangian graphs, then it is contained in some Mañé set and then is equal to some  $\mathcal{G}_c$ . We obtain then

that there is a partition of  $T^*M$  into some smooth  $\mathcal{G}_c$ . As  $(\mathcal{G}_c)_{c \in H^1(M, \mathbb{R})}$  is a partition of  $T^*M$ , we deduce that all the  $\mathcal{G}_c$  are smooth.

## 4.2 Proof of corollary 3

We just have to prove that  $2 \Rightarrow 1$ . We assume  $T^*M$  is covered by the union of its Lagrangian invariant smooth submanifolds that are Hamiltonianly isotopic to some smooth Lagrangian graph. We have proved in [3] a multidimensional Birkhoff theorem : every Lagrangian invariant smooth submanifold that is Hamiltonianly isotopic to some smooth Lagrangian graph is a smooth graph. Then corollary 3 becomes a corollary of corollary 2.



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